On Einstein's Theory of Gravitation, and its Astronomical Consequences. Third Paper.* By W. de Sitter, Assoc. R.A.S.

Contents of Third Paper.

1. In Einstein's theory of general relativity there is no essential difference between gravitation and inertia. The combined effect of the two is described by the fundamental tensor $g_{\mu\nu}$, and how much of it is to be called inertia and how much gravitation is entirely arbitrary. We might abolish one of the two words, and call the whole by one name only. Nevertheless it is convenient to continue to make a difference. Part of the $g_{\mu\nu}$ can be directly traced to the effect of known material bodies, and the common usage is to call this part "gravitation," and the rest "inertia." Then, if we take as a system of reference three rectangular cartesian space co-ordinates and the time multiplied by c (the velocity of light *in vacuo*), we know that, in that portion of the four-dimensional time-space which is accessible to our observations, the $g_{\mu\nu}$ of pure inertia are, within certain limits of uncertainty,

$$(1) \qquad \left\{\begin{array}{cccc} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{array}\right.$$

In our immediate neighbourhood, within the solar system, the limits of uncertainty are very narrow: say the eighth decimal

* See first paper, *M.N.*, vol. lxxvi. p. 699; second paper, *M.N.*, vol. lxxvii. p. 155. The present paper gives an account of the questions treated in the following communications:—

A. Einstein, "Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie," *Sitzungsber. Berlin*, 1917 Feb. 8, p. 142.

W. de Sitter, "On the Relativity of Inertia, remarks concerning Einstein's latest Hypothesis," *Proc. Akad. Amsterdam*, 1917 March 31, vol. xix. p. 1217.

W. de Sitter, "On the Curvature of Space," *Proc. Akad. Amsterdam*, 1917 June 30, vol. xx. (not yet published in English).

The notations used are the same as in the first and second papers. We may recall that $\delta_{\mu\mu}=1$, $\delta_{\mu\nu}=0$ for $\mu\neq\nu$, and that Σ is a sum from 1 to 4, and Σ' from 1 to 3.

place. As we get further away in space, or in time, or in both, the limits become wider: at a distance of a million light-years we can perhaps only guarantee the second decimal place.* How the $g_{\mu\nu}$ are in those portions of space and time to which our observations have not yet penetrated, we do not know, and how they are at infinity (of space or of time) we shall never know. All assumptions regarding the values of the $g_{\mu\nu}$ at infinity are therefore extrapolations, which we are free to choose in accordance with theoretical or philosophical requirements.

The extrapolation which most naturally offers itself, and which is also tacitly made in Newton's theory of inertia, is that the $g_{\mu\nu}$ retain the values (1) for all distances and times up to infinity. It has been pointed out in the second paper † that in this theory inertia is not relative. The values (1) are not invariant: the boundary-values of the $g_{\mu\nu}$ at infinity are different in different systems of co-ordinates. Einstein and others have therefore tried to find another extrapolation, by which the $g_{\mu\nu}$, while in our neighbourhood retaining the values (1) with the approximation demanded by the observations, would at infinity degenerate to a set of values which would be the same for all systems of reference.

The $g_{\mu\nu}$ are determined by the field-equations, which in Einstein's theory of 1915 are

(2) $$G_{\mu\nu} = -\kappa T_{\mu\nu} + \tfrac{1}{2}\kappa g_{\mu\nu} T,$$

or

(2′) $$G_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} G = -\kappa T_{\mu\nu},$$

and

$$G = \kappa T.$$

Once the system of reference of space- and time-variables has been chosen, these equations determine the $g_{\mu\nu}$ apart from constants of integration, or boundary-conditions at infinity. Only the deviations of the actual $g_{\mu\nu}$ from these values at infinity are thus due to the effect of matter, through the mechanism of the equations (2) or (2′). If at infinity all $g_{\mu\nu}$ were *zero*, then we could truly say that the *whole* of inertia, as well as gravitation, is thus produced. This is

* There are two criteria by which we can judge the value of the fundamental tensor at great distances from us. The frequency of light-vibrations is proportional to $\sqrt{g_{44}}$. Consequently, objects in whose spectra we are able to identify definite spectral lines must be situated in a portion of space where g_{44} is still of the order of unity. The motion of material particles, on the other hand, depends on all $g_{\mu\nu}$. We know that the relative velocities of the fixed stars are small. From this we conclude that also the accelerations are small. Let the velocities be of the order α, and let g_{44} be of the order γ, and $g_{ij} + \delta_{ij}$ of the order $\beta (i, j = 1, 2, 3)$. Then the accelerations contain terms of the order γ, γ^2, $\beta . \gamma$, $\alpha^2 . \gamma$, $\alpha^2 . \beta$, etc., but none of the order β. Thus here also we can only be sure of the smallness of γ, and not of β. Within the solar system the case is different, for there we have not only a statistical knowledge of the velocities, but we know the accelerations themselves; and our observations are so exact as to carry us to quantities of the second order. Consequently, we can be sure of g_{ij} to the first order, and of g_{44} to the second, the first order corresponding to about 10^{-8}.

† *M.N.*, vol. lxxvii. pp. 181–183.

the reasoning which has led to the postulate that at infinity all $g_{\mu\nu}$ shall be zero. I have called this the *mathematical postulate of relativity of inertia.*

If all matter were destroyed, with the exception of one material particle, then would this particle have inertia or not? The school of Mach requires the answer *No.* If, however, by "all matter" is meant all matter known to us, stars, nebulæ, clusters, etc., then the observations very decidedly give the answer *Yes.* The followers of Mach * are therefore compelled to assume the existence of still more matter. This matter, however, fulfils no other purpose than to enable us to suppose it not to exist, and to assert that in that case there would be no inertia. This point of view, which denies the logical possibility of the existence of a world without matter, I call the *material postulate of relativity of inertia.* The hypothetical matter introduced in accordance with it I call *world-matter.* Einstein originally supposed that the desired effect could be brought about by very large masses at very large distances. He has, however, now convinced himself that this is not possible. In the solution which he now proposes, the world-matter is not accumulated at the boundary of the universe, but distributed over the whole world, which is finite, though unlimited. Its density (in natural measure) is constant, when sufficiently large units of space are used to measure it. Locally its distribution may be very unhomogeneous. In fact, there is no essential difference between the nature of ordinary gravitating matter and the world-matter. Ordinary matter, the sun, stars, etc., are only condensed world-matter, and it is possible, though not necessary, to assume all world-matter to be so condensed. In this theory "inertia" is produced by the whole of the world-matter, and "gravitation" by its local deviations from homogeneity.

In Einstein's new solution the three-dimensional world is not infinite, but spherical.† Thus no boundary conditions at infinity are required. From the point of view of the theory of relativity it seems at first sight to be incorrect to say: the world *is* finite, since by a transformation of co-ordinates it can be made infinite, euclidean, or hyperbolical. Such transformations, however, leave the invariant G unaltered, and consequently also after the introduction of euclidean or hyperbolical co-ordinates the world remains finite and spherical in natural measure. The length of the semi-axis of x_1 in natural measure is

$$L_1 = \int_0^\infty \sqrt{-g_{11}}\, dx_1.$$

If this is to be finite, it is necessary that g_{11} shall become zero for $x_1 = \infty$; and inversely, if g_{11} becomes zero of a sufficiently high order for $x_1 = \infty$, then L_1 is finite. It is thus evident that the

* Mach himself still thought that the fixed stars would be sufficient. This, however, is not so.

† Or elliptical, see below, art. 2.

condition that the $g_{\mu\nu}$ shall be zero at infinity is equivalent to the finiteness of the world in natural measure.

It is found, however, that the $g_{\mu\nu}$ of this finite world do not satisfy the equations (2). Einstein is thus compelled to add a new term to these equations, which then become

(3) $$G_{\mu\nu} - \lambda g_{\mu\nu} = -\kappa T_{\mu\nu} + \tfrac{1}{2} g_{\mu\nu} T,$$

or

(3′) $$G_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}(G - 2\lambda) = -\kappa T_{\mu\nu};$$

from which we find easily

(4) $$G - 4\lambda = \kappa T.$$

If we put

$$G_{\mu\nu}' = G_{\mu\nu} - \lambda g_{\mu\nu},$$

we have

$$G' = G - 4\lambda.$$

Therefore the equations (3) and (3′) are found if in (2) or (2′) we replace $G_{\mu\nu}$ and G by $G_{\mu\nu}'$ and G′. Consequently the equations (3) can be derived from the generalised principle of Hamilton,* if we now take

$$H_3 = \int \sqrt{-g}(G - 4\lambda) d\tau.$$

All the conservative properties which follow from the principle of Hamilton thus remain true after the introduction of λ.

The curvature of the four-dimensional time-space is proportional to G. In the new theory we have $G = \kappa T + 4\lambda$: thus if there were no matter ($T = 0$), this curvature would not be zero.

Einstein's solution of the equations (3) implies the existence of a "world-matter" which fills the whole universe, as has already been mentioned. It is, however, also possible to satisfy the equations without this hypothetical world-matter. Then, of course, the "material postulate of relativity of inertia" is not satisfied, but the "mathematical postulate," which makes no mention of matter, but only requires the $g_{\mu\nu}$ to be zero at infinity, is satisfied. This is brought about by the introduction of the term with λ, and not by the world-matter, which, from this point of view, is not essential.

If we neglect all pressures and other internal forces, and if we suppose all matter to be at rest, then the tensor $T_{\mu\nu}$ becomes

(5) $$T_{44} = g_{44}\rho, \qquad \text{all other } T_{\mu\nu} = 0,$$

ρ being the density in natural measure. We can put

(6) $$\rho = \rho_0 + \rho_1,$$

where ρ_0 is the average density of the world-matter. If ρ_0 is positive, then ρ_1 may be positive or negative; but in the latter case the numerical value must not exceed ρ_0.

* See first paper, *M.N.*, lxxvi. p. 707.

If we wish to neglect gravitation, we must neglect ρ_1, and take ρ_0 constant. The equations (3) then become*

$$(7) \qquad \begin{cases} G_{ij} - (\lambda + \tfrac{1}{2}\kappa\rho_0)g_{ij} = 0. \\ G_{44} - (\lambda + \tfrac{1}{2}\kappa\rho_0)g_{44} = -\kappa\rho_0 g_{44}. \end{cases}$$

These can be satisfied by the $g_{\mu\nu}$ implied by the line-element

$$(8\text{A}) \qquad ds^2 = -dr^2 - R^2 \sin^2\frac{r}{R}[d\psi^2 + \sin^2\psi d\theta^2] + c^2dt^2,$$

if

$$(9\text{A}) \qquad \kappa\rho_0 = 2\lambda, \qquad \lambda = \frac{1}{R^2}.$$

This is Einstein's new solution.

The equations are also satisfied by

$$(8\text{B}) \qquad ds^2 = -dr^2 - R^2 \sin^2\frac{r}{R}[d\psi^2 + \sin^2\psi d\theta^2] + \cos^2\frac{r}{R}c^2dt^2,$$

if

$$(9\text{B}) \qquad \rho_0 = 0, \qquad \lambda = \frac{3}{R^2};$$

and, of course, also by

$$(8\text{C}) \qquad ds^2 = -dr^2 - r^2[d\psi^2 + \sin^2\psi d\theta^2] + c^2dt^2,$$

with

$$(9\text{C}) \qquad \rho_0 = 0, \qquad \lambda = 0.$$

This last solution (C) gives the $g_{\mu\nu}$ of the old theory of relativity, or of Newton's theory of inertia. In it three-dimensional space is euclidean, in (A) and (B) it has a constant positive curvature. In (A) there is a world-matter; in (B) and (C) we have $\rho_0 = 0$: the hypothetical world-matter does not exist.

2. If in (8A) and (8B) we put

$$(10) \qquad r = R\chi,$$

the three-dimensional line-element becomes

$$(11) \qquad d\sigma^2 = R^2\{d\chi^2 + \sin^2\chi[d\psi^2 + \sin^2\psi d\theta^2]\}.$$

This is the line-element of a three-dimensional space with a constant positive curvature, which is

$$\epsilon = \frac{1}{R^2}.$$

There are two possible forms of space with constant positive curvature, viz. the spherical space, or space of Riemann,† and the elliptical space, which has been investigated by Newcomb.‡ In the spherical space all straight lines starting from a point intersect

* The equations will be further developed in art. 5, below.

† "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen," *Werke*, p. 272.

‡ "Elementary theorems relating to the geometry of a space of three dimensions and of uniform positive curvature in the fourth dimension," *Crelles Journal*, vol. lxxxiii. p. 293.

again in the "antipodal" point, whose distance from the first point measured along any of these lines is πR. In the elliptical space any two straight lines cannot have more than one point in common. In both forms of space the straight line is closed: its total length is $2\pi R$ in the spherical space, and πR in the elliptical space. In the spherical space the largest possible distance between two points is πR, and there is only one point, the "antipodal point," at that distance from a given point. In the elliptical space the largest possible distance is $\frac{1}{2}\pi R$, and all points at that distance from a given point lie on a straight line—the "polar line" of the point. Both spaces are finite. The total volume of the spherical space is $2\pi^2 R^3$, and of the elliptical $\pi^2 R^3$.

Einstein only mentions the spherical space, which by the two-dimensional analogy of the sphere is easier to represent to our imagination. The elliptical space is, however, really the simpler case, and it is preferable to adopt this for the physical world.* Also the spherical space would give rise to difficulties, which will be pointed out below.

We can, instead of the co-ordinates r, ψ, θ, introduce other co-ordinates by which the elliptical, or spherical, space is projected on an euclidean or on a hyperbolical space. By the transformation

$$r = R \tan \chi \tag{12}$$

the whole of the elliptical space is projected on the whole of the euclidean space.† The projection of the spherical space fills the

* This is also the opinion of Einstein (communicated to the writer by letter).

† By the transformation

$$r_1 = R \sin \chi$$

the elliptical space is made to correspond with the inside of the sphere $r_1 \leq R$ in the euclidean space. The representation of the spherical space fills this sphere twice. If we put

$$x_1 = r_1 \sin\psi \sin\theta,$$
$$y_1 = r_1 \sin\psi \cos\theta,$$
$$z_1 = r_1 \cos\psi,$$

the co-ordinates x_1 y_1, z_1 are those used by Einstein in his paper of 1917 Feb. In these co-ordinates the three-dimensional line-element is

$$d\sigma^2 = \sum_i{}' dx_i^2 + \sum_i{}'\sum_j{}' \frac{x_i x_j dx_i dx_j}{R^2 - r_1^2}.$$

If we add

$$u_1 = R \cos \chi,$$

then x_1, y_1, z_1, u_1 are the co-ordinates used by Weierstrasz.

Riemann used the co-ordinates found by the transformation

$$r_2 = 2R \tan \tfrac{1}{2}\chi.$$

The line-element then becomes

$$d\sigma^2 = \frac{dr_2^2 + r_2^2[d\psi^2 + \sin^2\psi d\theta^2]}{(1 + r_2^2/4R^2)^2}.$$

By this transformation (which was also used in the paper by the writer of 1917 March) the whole of the spherical space corresponds to the whole of the euclidean space. The elliptical space corresponds to the inside of the sphere $r_2 \leq 2R$.

The transformation used in the text leads to the co-ordinates of Beltrami.

euclidean space twice, the projections of antipodal points being the same.

The four-dimensional line-element in these co-ordinates is, for the two systems,

$$(13A) \qquad ds^2 = -\frac{dr^2}{(1+\epsilon r^2)^2} - \frac{r^2[d\psi^2 + \sin^2\psi d\theta^2]}{1+\epsilon r^2} + c^2dt^2,$$

$$(13B) \qquad ds^2 = -\frac{dr^2}{(1+\epsilon r^2)^2} - \frac{r^2[d\psi^2 + \sin^2\psi d\theta^2]}{1+\epsilon r^2} + \frac{c^2dt^2}{1+\epsilon r^2}.$$

If now we put

$$x_1 = r \sin\psi \sin\theta$$
$$x_2 = r \sin\psi \cos\theta$$
$$x_3 = r \cos\psi$$
$$x_4 = ct$$

then the $g_{\mu\nu}$ for these co-ordinates are

$$g_{ij} = -\frac{\delta_{ij}}{1+\epsilon r^2} + \frac{\epsilon x_i x_j}{(1+\epsilon r^2)^2}, \qquad \begin{cases} \text{A.} & g_{44} = 1. \\ \text{B.} & g_{44} = \dfrac{1}{1+\epsilon r^2}. \end{cases}$$

The $g_{\mu\nu}$ for $r = 0$ have the values (1) in both systems A and B. For $r = \infty$ they degenerate to

$$(1A) \qquad \begin{Bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix}$$

$$(1B) \qquad \begin{Bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Bmatrix}$$

The set (1A) is invariant for all transformations for which (at infinity) $t' = t$; the set (1B) is invariant for *all* transformations.* It thus appears that the system A only satisfies the mathematical postulate of relativity if the latter is applied to three-dimensional space only. In other words, if we conceive the three-dimensional space (x_1, x_2, x_3) with its world-matter as movable in an absolute space, its movements can never be detected by observations: all motions of material bodies are relative to the space (x_1, x_2, x_3) with the world-matter, not to the absolute space. The world-matter thus takes the place of the absolute space in Newton's theory, or of the "inertial system." It is nothing else but this inertial system materialised. It should be pointed out that this

* With the restriction that none of the coefficients $\frac{dx_i}{dx_j'}$ becomes infinite at infinity.

relativity of inertia is in the system A only realised by making the time practically absolute. It is true that the fundamental equations of the theory, the field-equations (3) and the equations of motion, *i.e.* the differential equations of the geodetic line, remain invariant for all transformations. But only such transformations for which at infinity $t'=t$ can be carried out without altering the values (1A). In the system B, on the other hand, there is complete invariance for all transformations involving the four variables.

The system B is the four-dimensional analogy of the three-dimensional space of the system A. If we put

$$(14) \quad ds^2 = -R^2\{d\omega^2 + \sin^2\omega(d\zeta^2 + \sin^2\zeta[d\psi^2 + \sin^2\psi d\theta^2])\},$$

the $g_{\mu\nu}$ implied by this line-element satisfy the equations (3), with the conditions (9B). In order to avoid imaginary angles, we can put

$$\omega = i\omega', \qquad \zeta = i\zeta'.$$

Then the line-element becomes *

$$(15) \quad ds^2 = R^2\{d\omega'^2 - \sinh^2\omega'(d\zeta'^2 + \sinh^2\zeta'[d\psi^2 + \sin^2\psi d\theta^2])\}.$$

If now we put

$$\rho = R\tanh\omega'\sinh\zeta',$$
$$\tau = R\tanh\omega'\cosh\zeta',$$

then we have

$$(16) \quad ds^2 = \frac{-(1-\epsilon\tau^2)d\rho^2 - 2\epsilon\rho\tau d\rho d\tau + (1+\epsilon\rho^2)d\tau^2}{[1+\epsilon(\rho^2-\tau^2)]^2} - \frac{\rho^2[d\psi^2 + \sin^2\psi d\tau^2]}{1+\epsilon(\rho^2-\tau^2)}.$$

* If we take

$$r = R\sinh\omega'\sinh\zeta', \qquad t = R\sinh\omega'\cosh\zeta',$$
$$x = r\sin\psi\sin\theta,$$
$$y = r\sin\psi\cos\theta, \qquad u = R\cosh\omega',$$
$$z = r\cos\psi,$$

we have

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2 - du^2,$$

and

$$(a) \qquad R^2 - x^2 - y^2 - z^2 + t^2 - u^2 = 0.$$

The latter equation represents an hyperboloid (one-bladed) in the five-dimensional space (x, y, z, t, u). The projection of a point x, y, z, t, u of this hyperboloid from the point $x=y=z=t=u=0$ on the four-dimensional space $u=R$ has the co-ordinates (ξ, η, ζ, τ), where

$$\xi = \rho\sin\psi\sin\theta,$$
$$\eta = \rho\sin\psi\cos\theta,$$
$$\zeta = \rho\cos\psi.$$

This projection is limited by the "hyperbola"

$$(b) \qquad R^2 + \xi^2 + \eta^2 + \zeta^2 - \tau^2 = 0, \quad \text{or} \quad 1 + \epsilon(\rho^2 - \tau^2) = 0,$$

which is the projection of the points at infinity on the hyperboloid (*a*). The part of $u=R$ which is outside the hyperbola (*b*) is the projection of the (two-bladed) hyperboloid which is conjugated to (*a*). It will be seen from (16) that on the limiting "hyperbola" (*b*) all $g_{\mu\nu}$ become infinite.

Finally, by the transformation

$$\mathrm{R}\sin\frac{r}{\mathrm{R}}=\frac{\rho}{\sqrt{1+\epsilon(\rho^2-\tau^2)}},\qquad \mathrm{R}\sinh\frac{ct}{\mathrm{R}}=\frac{\tau}{\sqrt{1-\epsilon\tau^2}},$$

we find the formula (8B).

In the three-dimensional space, whose line-element is (11), we can transfer the origin to a point $(\chi_1, \psi_1, \theta_1)$, and the line-element expressed in co-ordinates referred to this new origin will again have the same form (11). Exactly in the same way we can in (14) transfer the origin to a point $(\omega_1, \zeta_1, \psi_1, \theta_1)$, corresponding to $(\chi_1, \psi_1, \theta_1, ct_1)$ in (8B). The line-element in the co-ordinates referred to this new origin will again have the same form (14), and this can again be transformed to new variables $\chi', \psi', \theta', ct'$, and will then again have the form (8B). Of course ct' will generally be different from ct.

In both systems A and B it is always possible, at every point of the four-dimensional time-space, to find systems of reference in which the $g_{\mu\nu}$ depend only on one space-variable (the "radius-vector"), and not on the "time." In the system A the "time" of these systems of reference is the same always and everywhere, in B it is not. In B there is no universal time; there is no essential difference between the "time" and the other three co-ordinates. None of them has any real physical meaning. In A, on the other hand, the time is essentially different from the space-variables.

3. In order further to compare the two systems, we will consider the course of rays of light. In A, if we use the co-ordinates r, ψ, θ, ct, the velocity of light is constant, and the rays of light, which are geodetic lines in the four-dimensional time-space, are also geodetic in the three-dimensional space r, ψ, θ. On triangles formed by such lines the ordinary formulæ of spherical trigonometry are applicable. Thus, if we suppose the sun to be at rest in the origin of co-ordinates, and if the distance sun-earth be called a, then the parallax* p of a star whose distance from the sun is r, is given by

$$\tan p=\sin\frac{a}{\mathrm{R}}\cot\frac{r}{\mathrm{R}};$$

or, since the square of a/R can be neglected,

$$p=\frac{a}{\mathrm{R}}\cot\frac{r}{\mathrm{R}}. \tag{17}$$

The same result is found in the reference system r, ψ, θ, ct. By the transformation (12) all straight lines remain straight in the projection. We can, moreover, easily verify that the rays of

* The parallax is 90° − A, if A is the angle at the earth, the angle at the sun being 90°. In spherical geometry, of course, 90° − A is *not* equal to the angle at the star, as it is in euclidean geometry.

light must be straight lines in the system r, ψ, θ, ct. The velocity of light in this system is

$$v = \frac{c(1 + \epsilon r^2)}{\sqrt{(1 + \epsilon r^2 \sin^2 V)}},$$

where V is the angle between the radius-vector and the tangent to the ray of light. The equation of the ray of light then becomes *

$$\sin V = \frac{k}{r},$$

k being a constant. This is the equation of a straight line. The parallax is thus determined by the ordinary formulas of euclidean geometry, and we have

$$p = \frac{a}{r} = \frac{a}{R} \cot \frac{r}{R},$$

which is the same as (17).

The parallax vanishes for $r = \frac{1}{2}\pi R$, *i.e.* for the largest distance which can occur in elliptical space. If we adopted the spherical space, so that still larger distances could occur, p would become negative, and for $r = \pi R$ we would have $p = -90°$.

In the system B the rays of light are *not* geodetic lines in the three-dimensional space (r, ψ, θ), nor in (r, ψ, θ). In (r, ψ, θ) the velocity of light is $v = c \cos \chi$. If now we introduce a new variable h by the condition

$$\frac{dr}{dh} = \cos \chi,$$

of which the integral is

$$\sinh \frac{h}{R} = \tan \frac{r}{R} = \frac{r}{R}, \qquad (18)$$

then the velocity of light in the radial direction will be constant. The line-element becomes †

$$ds^2 = \frac{-dh^2 - R^2 \sinh^2 \frac{h}{R}[d\psi^2 + \sin^2 \psi d\theta^2] + c^2dt^2}{\cosh^2 \frac{h}{R}}. \qquad (19B)$$

* See first paper, *M.N.*, lxxvi. p. 717.

† The transformation (18) can, of course, also be applied in the system A. Then the line-element becomes

$$ds^2 = \frac{-dh^2 - R^2 \sinh^2 \frac{h}{R}[d\psi^2 + \sin^2 \psi d\theta^2]}{\cosh^2 \frac{h}{R}} + c^2dt^2. \qquad (19A)$$

In (19A) all g^{ij} become zero for $h = \infty$, but g_{44} remains 1; in (19B) g_{44} also becomes zero.

The three-dimensional space of this system of reference is the space with constant *negative* curvature, or hyperbolical space, or space of Lobatschewski. It is evident from (18) that the whole of elliptical space corresponds to the whole of the hyperbolical space; the representation of the spherical space would fill the hyperbolical space twice.

In the system of reference h, ψ, θ, ct the velocity of light is constant [in all directions, though the transformation (18) was found from the condition that it should be constant in the radial direction], and the rays of light are straight (*i.e.* geodetic) lines in the three-dimensional hyperbolical space (h, ψ, θ). This hyperbolical space thus in the system B plays the same part as the elliptical space in the system A (and the euclidean space in the system C), so far as the propagation of light is concerned. If the motion of material particles (mechanics) also is considered, then the analogy breaks down, owing to the numerator $\cosh^2 h/\mathrm{R}$.

The light-rays being straight lines, we can for the derivation of the parallax use the formulas of trigonometry in hyperbolical geometry. We thus find

$$\tan p = \sinh \frac{a}{\mathrm{R}} \coth \frac{h}{\mathrm{R}},$$

or

$$p = \frac{a}{\mathrm{R}} \coth \frac{h}{\mathrm{R}}. \tag{20}$$

It follows that in the system B the parallax of a star can never be zero. For $h = \infty$ we have $p = \dfrac{a}{\mathrm{R}}$. By the transformation (18) we have

$$p = \frac{a}{\mathrm{R} \sin \chi} = \frac{a}{\mathrm{r}} \sqrt{1 + \frac{\mathrm{r}^2}{\mathrm{R}^2}}. \tag{20'}$$

Thus p reaches its minimum value $\dfrac{a}{\mathrm{R}}$ for $\chi = \frac{1}{2}\pi$. For larger values of χ, which can only occur in the spherical space, p would increase again, and for $r = \pi \mathrm{R}$ we would have $p = 90^\circ$. In fact, if the spherical space is projected by (18) on the hyperbolical space, the projections of antipodal points coincide: a star at the antipodal point of the sun would be projected in the sun.

It may be interesting to derive the formula (20′) from the course of rays of light in the system (r, ψ, θ). In this system the velocity of light is

$$v = c\sqrt{\frac{1 + \epsilon \mathrm{r}^2}{1 + \epsilon \mathrm{r}^2 \sin^2 \mathrm{V}}}.$$

The equation of the ray of light becomes

$$\sin \mathrm{V} = \frac{a}{\mathrm{r}(1 + \epsilon \mathrm{r}^2)}.$$

The parallax is determined by the equation *

$$\frac{dp}{d\mathrm{r}} = -\frac{\tan \mathrm{V}}{\mathrm{r}},$$

from which, if we neglect a^2/r^2, we find

$$p = \frac{a}{\mathrm{r}}\sqrt{1+\epsilon\mathrm{r}^2},$$

which is the same as (20′).

4. The equations of motion of a material particle in the field of pure inertia are the differential equations of the geodetic line, viz.

$$(21) \qquad \frac{d^2x_i}{c^2dt^2} = -\sum_p \sum_q \left[\begin{Bmatrix} pq \\ i \end{Bmatrix} - \begin{Bmatrix} pq \\ 4 \end{Bmatrix}\dot{x}_i\right]\dot{x}_p\dot{x}_q ;$$

or, if we restrict ourselves to such systems of reference in which the $g_{\mu\nu}$ do not depend on $x_4 = ct$,

$$(21') \quad \frac{d^2x_i}{c^2dt^2} = -\begin{Bmatrix} 44 \\ i \end{Bmatrix} - \sum_p{}' \sum_q{}' \begin{Bmatrix} pq \\ i \end{Bmatrix}\dot{x}_p\dot{x}_q + 2\sum_p{}' \begin{Bmatrix} p4 \\ 4 \end{Bmatrix}\dot{x}_i\dot{x}_p.$$

In the system C, which represents Newton's theory of inertia, if we take rectangular cartesian space-co-ordinates, the g_{ij} are given by (1), and all the brackets are zero. Consequently

$$\frac{d^2x_i}{c^2dt^2} = 0.$$

The orbit of a particle under the influence of inertia without gravitation is thus a straight line in euclidean space, and the velocity is constant.

In the system A we have for the co-ordinates r, ψ, θ, ct,

$$\begin{Bmatrix} 22 \\ 1 \end{Bmatrix} = -\mathrm{R}\sin\chi\cos\chi, \qquad \begin{Bmatrix} 33 \\ 1 \end{Bmatrix} = -\mathrm{R}\sin\chi\cos\chi\sin^2\psi,$$

$$\begin{Bmatrix} 12 \\ 2 \end{Bmatrix} = \begin{Bmatrix} 13 \\ 3 \end{Bmatrix} = \frac{1}{\mathrm{R}}\cot\chi, \quad \begin{Bmatrix} 33 \\ 2 \end{Bmatrix} = -\sin\psi\cos\psi, \quad \begin{Bmatrix} 23 \\ 3 \end{Bmatrix} = \cot\psi.$$

* Strictly speaking, r here is the distance from the star to the earth, instead of to the sun. The square of a^2/r^2 being neglected, these two distances may be interchanged. We thus have, in the notation of the first paper,

$$p = \mathrm{V} - x.$$

Now we have (see first paper, p. 718)

$$\frac{dx}{d\mathrm{V}} = 1 + \frac{\tan\mathrm{V}}{\mathrm{r}}\frac{d\mathrm{r}}{d\mathrm{V}},$$

or

$$\frac{dx}{d\mathrm{r}} = \frac{d\mathrm{V}}{d\mathrm{r}} + \frac{\tan\mathrm{V}}{\mathrm{r}},$$

from which the equation for p follows immediately.

The other brackets are zero. We find:

$$\frac{d^2r}{c^2dt^2} = R \sin\chi \cos\chi \left[\left(\frac{d\psi}{cdt}\right)^2 + \sin^2\psi\left(\frac{d\theta}{cdt}\right)^2\right],$$

$$\frac{d^2\theta}{c^2dt^2} = -\frac{2}{R}\cot\chi\frac{dr}{cdt}\cdot\frac{d\theta}{cdt} - 2\cot\psi\frac{d\psi}{cdt}\cdot\frac{d\theta}{cdt},$$

$$\frac{d^2\psi}{c^2dt^2} = -\frac{2}{R}\cot\chi\frac{dr}{cdt}\cdot\frac{d\psi}{cdt} + \sin\psi\cos\psi\left(\frac{d\theta}{cdt}\right)^2.$$

We can take $\psi = 90°$, $\frac{d\psi}{dt} = 0$. Then we find the integrals of areas and of living force:

$$(22) \qquad \begin{cases} R^2\sin^2\chi\left(\frac{d\theta}{dt}\right) = c \\ R^2\sin^2\chi\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2 = k. \end{cases}$$

Eliminating dt, we find the differential equation of the orbit

$$(23) \qquad \left(\frac{dr}{d\theta}\right)^2 + R^2\sin^2\chi = \frac{k}{c^2}R^4\sin^4\chi.$$

The integral is

$$(24) \qquad \tan\chi\cos(\theta - \theta_0) = \frac{c}{kR^2 - c}.$$

This is the equation of a straight (*i.e.* geodetic) line in the spherical or elliptical space. By the second of (22) the velocity is constant. Thus in the system A a material particle under the action of inertia alone describes a straight line in elliptical space with a constant velocity.

In the case of the system B we will use the co-ordinates r, ψ, θ, ct. Then we have:

$$\begin{Bmatrix}11\\1\end{Bmatrix} = -\frac{2\epsilon r}{1+\epsilon r^2}, \quad \begin{Bmatrix}22\\1\end{Bmatrix} = -r, \quad \begin{Bmatrix}33\\1\end{Bmatrix} = -r\sin^2\psi, \quad \begin{Bmatrix}44\\1\end{Bmatrix} = -\epsilon r,$$

$$\begin{Bmatrix}12\\2\end{Bmatrix} = \begin{Bmatrix}13\\3\end{Bmatrix} = \frac{1}{r}, \quad \begin{Bmatrix}33\\2\end{Bmatrix} = -\sin\psi\cos\psi, \quad \begin{Bmatrix}23\\3\end{Bmatrix} = \cot\psi, \quad \begin{Bmatrix}14\\4\end{Bmatrix} = -\frac{\epsilon r}{1+\epsilon r^2}.$$

The others are zero. We find now:

$$\frac{d^2r}{c^2dt^2} = \epsilon r + r\left[\left(\frac{d\psi}{cdt}\right)^2 + \sin^2\psi\left(\frac{d\theta}{cdt}\right)^2\right],$$

$$\frac{d^2\theta}{c^2dt^2} = -\frac{2}{r}\frac{dr}{cdt}\frac{d\theta}{cdt} - 2\cot\psi\frac{d\psi}{cdt}\frac{d\theta}{cdt},$$

$$\frac{d^2\psi}{c^2dt^2} = -\frac{2}{r}\frac{dr}{cdt}\frac{d\psi}{cdt} + \sin\psi\cos\psi\left(\frac{d\theta}{cdt}\right)^2.$$

We can again take $\psi = 90°$, $\frac{d\psi}{dt} = 0$. The integrals of areas and living force are

$$(25) \qquad \begin{cases} r^2 \frac{d\theta}{dt} = c. \\ \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = \epsilon r^2 + k. \end{cases}$$

The differential equation of the orbit is

$$(26) \qquad \left(\frac{dr}{d\theta}\right)^2 + r^2 = \frac{\epsilon r^2 + k}{c^2} . r^4.$$

The integration is easily effected by putting $y = \frac{1}{2r^2}$. We find

$$(27) \qquad r^2[1 + e \cos 2(\theta - \theta_0)] = \frac{2c^2}{k},$$

where

$$e = \frac{\sqrt{(4\epsilon c^2 + k^2)}}{k}.$$

This becomes a straight line in elliptical space * only if $e = 1$ or $c = 0$, *i.e.* $\frac{d\theta}{dt} = 0$. The orbit is thus only straight if it passes through the origin.

We can complete the integration by introducing an auxiliary angle u. We find the formulas

$$(28) \qquad \begin{cases} r^2 = \frac{1}{2}R^2k(e \cosh 2u - 1), \\ r^2 \cos 2(\theta - \theta_0) = \frac{1}{2}R^2k(e - \cosh 2u), \\ r^2 \sin 2(\theta - \theta_0) = \frac{1}{2}R^2k\sqrt{e^2 - 1} \sinh 2u, \\ \tan (\theta - \theta_0) = \sqrt{\frac{e+1}{e-1}} \tanh u, \\ u = \frac{t}{R} + u_0. \end{cases}$$

We have

$$\frac{dr}{dt} = \cos^2 \chi \frac{dr}{dt}.$$

* In the co-ordinates h, ψ, θ (hyperbolical space) the equation (27) becomes

$$R^2 \tanh^2 \frac{h}{R}[k + 2\epsilon c^2 + ke \cos 2(\theta - \theta_0)] = 2c^2,$$

which is a straight line when

$$e = 1 + \frac{2\epsilon c^2}{k}.$$

Consequently, the integrals (25) expressed in the co-ordinates r, ψ, θ of elliptical space become *

$$(25') \quad \begin{cases} R^2 \tan^2 \chi \dfrac{d\theta}{dt} = c. \\ \left(\dfrac{dr}{dt}\right)^2 + R^2 \sin^2 \chi \left(\dfrac{d\theta}{dt}\right)^2 = \sin^2 \chi \cos^2 \chi + (k + \epsilon c^2) \cos^4 \chi. \end{cases}$$

In the system B, therefore, a material particle under the influence of inertia alone does *not* describe a straight line with constant velocity. The orbit can only be straight if it passes through the origin, but even then the velocity is not constant. For small values of χ the equations (25′) do, however, not differ from (22). Those parts of the orbit which come within the reach of our observations therefore are sensibly straight, if we adopt a sufficiently large value of R.

The velocity becomes zero for $r = \frac{1}{2}\pi R$. Thus a material particle which is on the polar line of the origin can have no velocity. It also has no energy, for the energy of a material particle is

$$m \sum_p g_{p4} \frac{dx_p}{ds},$$

which also vanishes for $r = \frac{1}{2}\pi R$. Also the velocity of light is zero on the polar line.

All these results sound very strange and paradoxical. They are, of course, all due to the fact that g_{44} becomes zero for $r = \frac{1}{2}\pi R$. We can say that on the polar line the four-dimensional time-space is reduced to the three-dimensional space: *there is no time*, and consequently no motion.

It may be pointed out that the time taken by light to reach the distance $\frac{1}{2}\pi R$ from the origin (or from any other point) is

$$T = \frac{R}{c} \int_0^{\frac{1}{2}\pi} \sec \chi d\chi = \infty.$$

A fortiori the time needed by a material particle for the same journey is also infinite. This also follows from the equations (28), for the distance $r = \frac{1}{2}\pi R$ corresponds to $r = \infty$, and consequently to $u = \pm \infty$, or $t = \pm \infty$. A particle which has not always been on the polar line can therefore only reach it after an infinite time, *i.e.* it can never reach it at all. We can thus say that all the paradoxical phenomena (or rather negations of phenomena) which

* In the co-ordinates of hyperbolical space we have

$$R^2 \sinh^2 \frac{h}{R} \frac{d\theta}{dt} = c,$$

$$\left(\frac{dh}{dt}\right)^2 + R^2 \sinh^2 \frac{h}{R} \left(\frac{d\theta}{dt}\right)^2 = \tanh^2 \frac{h}{R} + (k + \epsilon c^2) \operatorname{sech}^2 \frac{h}{R}.$$

The law of areas is therefore true in hyperbolical space in the system B, as it is in elliptical space in the system A, and in euclidean space in C.

have been enumerated above can only happen after the end or before the beginning of eternity.*

Of course such things as "velocity" and "energy" are relative to the system of co-ordinates. They are not tensors, and consequently different in different systems of reference. It may well be that the system r, ψ, θ, ct is not the most simple or the most convenient to describe the phenomena. When described in other co-ordinates the same results may present themselves in a different form. But the fact remains that the extrapolation according to the hypothesis B is more different from what we are used to in our neighbourhood than that according to the hypotheses A or C.

The system A satisfies the "material postulate of relativity of inertia," but it restricts the admissible transformations to those for which at infinity $t' = t$, and thus introduces a quasi-absolute time, as has been explained in art. 2. In B and C the time is entirely relative, and completely equivalent to the other three co-ordinates. In A there is a world-matter, with which the whole world is filled, and this can be in a state of equilibrium without any internal stresses or pressures, if it is entirely homogeneous and at rest. In B there may, or may not, be matter, but if there is more than one material particle these cannot be at rest, and if the whole world were filled homogeneously with matter this could not be at rest without internal pressure or stress; for if it were, we would have the system A, with $g_{44} = 1$ for all values of the four co-ordinates. The system B satisfies the "mathematical postulate" of relativity of inertia, which does not appear to admit of a simple physical interpretation.

In the system C we have no relativity of inertia at all. It cannot be denied that the introduction of the constant λ, which distinguishes the systems A and B from C, is somewhat artificial, and detracts from the simplicity and elegance of the original theory of 1915, one of whose great charms was that it embraced so much without introducing any new empirical constant.

Postscriptum to Art. 4 (*added* 1917 *October*).

[The orbit of a material particle in the system B under the influence of inertia alone is given by the equation (27). This equation represents a *hyperbola*. If by r_0 we denote the minimum value of r, and by v_0 the velocity ($d\mathrm{r}/cdt$) at this point, then we have

$$e = \frac{\mathrm{v}_0^2 + \epsilon \mathrm{r}_0^2}{\mathrm{v}_0^2 - \epsilon \mathrm{r}_0^2}, \qquad \mathrm{c} = \mathrm{r}_0 \mathrm{v}_0, \qquad \mathrm{k} = \mathrm{v}_0^2 - \epsilon \mathrm{r}_0^2.$$

Further, if we put

$$\mathrm{x} = \mathrm{r} \cos(\theta - \theta_0), \qquad \mathrm{y} = \mathrm{r} \sin(\theta - \theta_0),$$

* In the systems of reference in which the radius-vector is measured by r (projection on euclidean space) and h (projection on hyperbolical space) they are also relegated to infinity of space.

then the equation (27) reduces to

$$(27') \qquad \frac{x^2}{r_0^2} - \frac{y^2}{R^2 v_0^2} = 1,$$

which represents a hyperbola of which the real axis is r_0 and the imaginary axis Rv_0. For a velocity of half a mile per day this latter axis still exceeds the distance of Neptune from the sun (assuming $R = 10^{12}$), and consequently for all observable phenomena the hyperbola can be treated as a straight line.

The equations (28) may be similarly transformed to

$$(28') \qquad x = r_0 \cosh u, \qquad y = Rv_0 \sinh u, \qquad u = \frac{t}{R} + u_0.$$

For $v_0 = 1$ the velocity is equal to the velocity of light, and the orbit becomes a ray of light. The rays of light are therefore (in the system of reference r, ψ, θ, ct) hyperbolas whose imaginary axis is R. It is easily verified that this is in accordance with the result found in art. 3.]

5. We will now further develop the field-equations

$$(3) \qquad G_{\mu\nu} - \lambda g_{\mu\nu} = -\kappa T_{\mu\nu} + \tfrac{1}{2}\kappa g_{\mu\nu} T.$$

We will consider no other source of gravitation than a material sphere at the origin of co-ordinates, which we will call the sun. In the systems B and C there is then no other matter than this: inside the sun we have $\rho = \rho_1$, and outside $\rho = 0$. In the system A the average density of the world-matter must remain constant. If part of it is condensed to form the sun, then the density in the neighbourhood of the sun is decreased, so that the total mass in a sufficiently large volume surrounding the sun is not affected. The mass of the sun is, however, extremely small compared with the total mass within one unit of volume of such size as must be used to measure the average density, and we can hardly postulate the total mass within each unit of volume to be *exactly* the same. We will therefore neglect the compensation, and take

within the sun: $\rho = \rho_0 + \rho_1$,
outside the sun: $\rho = \rho_0$.

Although this is not strictly in accordance with Einstein's hypothesis of constant average density, it is at all events a legitimate problem to investigate the field of gravitation and inertia for this distribution of matter.

We can take

$$ds^2 = -a dy^2 - b[d\psi^2 + \sin^2 \psi d\theta^2] + f c^2 dt^2,$$

where y represents any of the variables r, h, r, etc., which may be used to measure the radius-vector. The equations become somewhat simpler if we introduce

$$l = lg a, \qquad m = lg b, \qquad n = lg f.$$

We can suppose a, b, f to be functions of y and ct only. We indicate differential quotients with respect to y by accents, and with respect to ct by dots. Then we find

$$(29)\quad \left\{\begin{aligned} G_{11} &= m'' + \tfrac{1}{2}n'' + \tfrac{1}{2}m'(m'-l') + \tfrac{1}{4}n'(n'-l') \\ &\qquad - \frac{a}{f}\{\tfrac{1}{2}\ddot{l} + \tfrac{1}{4}\dot{l}(\dot{l} + 2\dot{m} - \dot{n})\}, \\ \frac{a}{b}G_{22} &= -\frac{a}{b} + \tfrac{1}{2}m'' + \tfrac{1}{4}m'(n' + 2m' - l') \\ &\qquad - \frac{a}{f}\{\tfrac{1}{2}\ddot{m} + \tfrac{1}{4}\dot{m}(\dot{l} + 2\dot{m} - \dot{n})\}, \\ -\frac{a}{f}G_{44} &= \tfrac{1}{2}n'' + \tfrac{1}{4}n'(n' + 2m' - l') \\ &\qquad - \frac{a}{f}\{\ddot{m} + \tfrac{1}{2}\ddot{l} + \tfrac{1}{2}\dot{m}(\dot{m} - \dot{n}) + \tfrac{1}{4}\dot{l}(\dot{l} - \dot{n})\}, \\ G_{33} &= \sin^2\psi \,.\, G_{22}. \end{aligned}\right.$$

We must now introduce the tensor $T_{\mu\nu}$. If we take the values (5), replacing ρ by ρ_0, the equations (3) become

$$G_{11} = -a(\lambda + \tfrac{1}{2}\kappa\rho_0),$$
$$\frac{a}{b}G_{22} = -a(\lambda + \tfrac{1}{2}\kappa\rho_0),$$
$$-\frac{a}{f}G_{44} = -a(\lambda - \tfrac{1}{2}\kappa\rho_0),$$

which are the equations (7) already given above. It is easily verified that all the different sets of $g_{\mu\nu}$ which have been given above for the inertial field satisfy these equations, if the appropriate values are taken for λ and ρ_0.

It has been found above that we can always introduce such a system of reference that a, b, f are functions of the variable y only. We can thus omit the lower lines of the expressions (29). Doing this, and using in $T_{\mu\nu}$ again ρ instead of ρ_0, I find by a slight transformation *

$$(30)\quad \left\{\begin{aligned} &(a)\quad n'' + n'(m' + \tfrac{1}{2}n' - \tfrac{1}{2}l') = a\kappa\rho - 2a\lambda, \\ &(b)\quad m'' + \tfrac{1}{2}m'(m' - n' - l') = -a\kappa\rho, \\ &(c)\quad -\frac{a}{b} + \tfrac{1}{2}m'(n' + \tfrac{1}{2}m') = -a\lambda. \end{aligned}\right.$$

In the equations (3′) the $g_{\mu\nu}$ occur not only in the left-hand members, but are also involved in the $T_{\mu\nu}$. In (30) we have taken the values (5) of $T_{\mu\nu}$. These correspond to the case that all matter is at rest, and not subject to any pressure or other internal forces. The fact that the $g_{\mu\nu}$ of the inertial field of the system A satisfies these equations thus proves that by inertia alone no internal stress

* We take

$$(a) = -2\frac{a}{f}G_{44}, \quad (b) = G_{11} + \frac{a}{f}G_{44}, \quad (c) = \frac{a}{b}G_{22} - \tfrac{1}{2}(b).$$

or pressure is produced in the world-matter, if this is at rest, and if ρ_0 is constant.

If the equations (30) are correct, which they only are in case the values (5) of the $T_{\mu\nu}$ are admissible, they must be dependent on each other.* If we form the combination

$$2\frac{d(c)}{dy}+2[m'-l'].(c)-[m'+n'].(b)-m'.(a),$$

then we find

$$(31) \qquad 0=a\kappa\rho n'.$$

The equations (30) are therefore only correct when either $\rho=0$ (as in the systems B and C outside matter), or $f=\text{const.}$ (as in A and C in the absence of gravitation). If the gravitational effect of the sun is not neglected, we cannot use the values (5) of $T_{\mu\nu}$. If in the system A we consider the world-matter as a continuous "fluid" at rest, there must be a stress or pressure in it; if it is considered as consisting of concrete material particles, these cannot be at rest. Which way of treating it is chosen, is not essential for our purpose. We will assume the world-matter to be an incompressible fluid. Then we have †

$$(32) \qquad T_{ii}=-g_{ii}p, \qquad T_{44}=g_{44}\rho.$$

If this is introduced, we find, instead of (30),

$$(33) \quad \begin{cases} (a) & n''+n'(m'+\tfrac{1}{2}n'-\tfrac{1}{2}l')=a\kappa(\rho+3p)-2a\lambda, \\ (b) & m''+\tfrac{1}{2}m'(m'-n'-l')=-a\kappa(\rho+p), \\ (c) & -\dfrac{a}{b}+\tfrac{1}{2}m'(n'+\tfrac{1}{2}m')=a\kappa p-a\lambda. \end{cases}$$

If p is determined in accordance with the principles of Einstein's theory, the equations (33) become dependent on each other.‡ We can therefore use the equations (33), with an arbitrary

* See first paper, art. 8 (*M.N.*, lxxvi. p. 708).

† See first paper, p. 713, where we put $P=0$.

‡ If for a, b, f we take the values of the inertial field of the system A, viz.

$$a=1, \qquad b=R^2\sin^2\chi, \qquad f=1,$$

then the three equations (33) are found to be dependent on each other, and to reduce to

$$(\alpha) \qquad \lambda=\frac{3}{R^2}-\kappa\rho_0, \qquad \kappa(p+\rho_0)=\frac{2}{R^2}.$$

If we take $p=0$, these give the values (9A) of ρ_0 and λ. If, however, we admit a pressure in the world-matter, we can have other values of ρ_0 and λ.

If we take the values of the inertial field of the system B, viz.

$$a=1, \qquad b=R^2\sin^2\chi, \qquad f=\cos^2\chi,$$

then the three equations are again dependent on each other, and reduce to

$$(\beta) \qquad \lambda=\frac{3}{R^2}-\kappa\rho_0, \qquad p+\rho_0=0.$$

For $p=0$ these give the values (9B), and, unless we are prepared to admit a negative pressure, this is the only solution.

These considerations originated from a remark made by Professor Lorentz additional condition, to determine a, b, f, and p. The same combination as was used above now gives, instead of (31),

$$(34) \qquad a\kappa[(\rho+p)n'+2p']=0.$$

We require only the field outside the sun. We can thus take $\rho=\rho_0$. Integrating (34), and determining the constant of integration from the condition that for $f=1$ we must have $p=0$, we find

$$(35) \qquad p=\rho_0\left(\frac{1}{\sqrt{f}}-1\right).$$

If this value of p is introduced, the equations (33) become dependent on each other. We can thus use two of them, and add an arbitrary condition. If we use the co-ordinates of elliptical space, *i.e.* if we take $y=r$, we can put

$$a=1+\alpha, \quad b=R^2\sin^2\chi(1+\beta), \quad f=1+\gamma.$$

Further, instead of (9A) we now take

$$\lambda=\epsilon(1+\zeta), \qquad \kappa\rho_0=2\lambda+\epsilon\,.\,\delta,$$

where ζ and δ are constants, which must be of the same order as α, β, γ. If we neglect quantities of the second order, we have from (35)

$$p=-\tfrac{1}{2}\rho_0\gamma.$$

The equations (33), of which we use the first and the last, then become, to the first order,

$$(36) \quad \begin{cases} (a) & R^2\gamma''+2R\cot\chi\,.\,\gamma'+3\gamma=\delta. \\ (c) & \beta\operatorname{cosec}^2\chi-\alpha\cot^2\chi+R\cot\chi(\beta'+\gamma')+\gamma+\zeta=0. \end{cases}$$

From (36, a) we find *

$$\gamma=-\mu\frac{\cos 2\chi}{\sin\chi}+\tfrac{1}{3}\delta.$$

For $\chi=\frac{1}{2}\pi$ we must have $\gamma=0$. This determines δ. We find $\delta=-3\mu$. The formula thus becomes

$$(37) \qquad \gamma=-\mu\left(\frac{\cos 2\chi}{\sin\chi}+1\right),$$

which, after development in powers of $1/R$, becomes

$$\gamma=-\mu R\left(\frac{1}{r}+\frac{1}{R}+\ldots\right).$$

(in a letter to the writer). Formulas equivalent to (a) also occur in a paper by Professor T. Levi-Civita, "Realta fisica di alcuni spazi normali di Bianchi" (*Rendiconti d. Acc. dei Lincei*, 1917 May, p. 521), which only came to my acquaintance after the present paper had been sent to the press.

* The complete integral has cos $(2\chi+\omega)$ instead of cos 2χ, ω being a constant of integration, for which we take the value zero.

It thus appears that μR is the same quantity that has been called $2\lambda_0{}^2$ in the first paper. If we take $R = 2 . 10^{12}$ (see art. 6, below), then for our sun we have $\mu = 10^{-20}$.

The remaining equation (36) is satisfied by

$$\alpha = \beta = -\gamma - \zeta.$$

The value of ζ is arbitrary. We can take $\zeta = 0$. We have thus, instead of (9A),

$$\lambda = \frac{1}{R^2}, \qquad \kappa\rho_0 = 2\lambda - \frac{3\mu}{R^2}.$$

For $r = \frac{1}{2}\pi R$ we have $\gamma = 0$; for larger values of r, which are only possible in spherical space, the numerical value of γ increases again; and for $r = 2\pi R$ we would have $\gamma = -\infty$, however small μ may be. In the elliptical space, of course, this difficulty does not exist, since distances exceeding $\frac{1}{2}\pi R$ are impossible.

In the system B we have, outside the sun, $\rho = 0$, and consequently also $p = 0$. The equations (33) then are dependent on each other. We now put

$$a = 1 + \alpha, \quad b = R^2 \sin^2\chi(1 + \beta), \quad f = \cos^2\chi(1 + \gamma),$$
$$\rho_0 = 0, \quad \lambda = 3\epsilon(1 + \zeta).$$

The equations become, to the first order,

$$(38) \quad \begin{cases} (a) & R^2\gamma'' + 2R\gamma'(\cot\chi - \tan\chi) + R\tan\chi(\alpha' - 2\beta') + 6(\alpha + \zeta) = 0, \\ (c) & (\beta - \alpha)\operatorname{cosec}^2\chi + 3(\alpha + \zeta) + R\beta'(\cot\chi - \tan\chi) + R\gamma'\cot\chi = 0. \end{cases}$$

These are satisfied by

$$\alpha = \frac{\mu}{\sin\chi} - \zeta,$$

$$\beta = -\frac{\mu}{\sin\chi} - \zeta,$$

$$\gamma = -\frac{\mu}{\sin\chi} + \zeta.$$

The value of ζ is irrelevant. We can take $\zeta = 0$. If we took $\zeta = \mu$, we would have $\alpha = \gamma = 0$ for $r = \frac{1}{2}\pi R$. If then we introduce

$$r' = r(1 - \tfrac{1}{2}\mu), \quad R' = R(1 - \tfrac{1}{2}\mu), \quad t' = t(1 + \tfrac{1}{2}\mu),$$

we have, neglecting the square of μ, for these new variables:

$$\chi = \frac{r'}{R'}, \qquad \lambda = \frac{3}{R'^2},$$

$$a' = 1 + \frac{\mu}{\sin\chi}, \qquad b' = R^2\sin^2\chi\left(1 - \frac{\mu}{\sin\chi}\right), \qquad f' = \cos^2\chi\left(1 - \frac{\mu}{\sin\chi}\right),$$

or the same formulas as for the unaccented letters with $\zeta = 0$.

6. We will now try to make some estimates of the value of R,

which must be adopted in order not to contradict the known data of observations. We will throughout use astronomical units: the unit of time being the day, of length the mean distance of the earth from the sun, and of mass the sun's mass.* In these units we have

$$c = 173, \quad k = \tfrac{1}{4} \,.\, 10^{-6}.$$

We will first take the system A.

In elliptical space the apparent angular diameter of an object whose linear diameter is d, at the distance $r = \mathrm{R}\chi$ from the earth, is

$$\delta = \frac{d}{\mathrm{R} \sin \chi}.$$

It is very probable that at least some of the spiral nebulæ or globular clusters are galactic systems comparable with our own in size. If then we take for the diameters, *e.g.*, $d = 10^9$ and $\delta = 5'$, then even for the maximum distance $r = \frac{1}{2}\pi\mathrm{R}$ we would have $\mathrm{R} = 6 \,.\, 10^{11}$. We are thus led to take, roughly,

(39) $$\mathrm{R} \geqq 10^{12}.$$

The total volume of elliptical space is $\mathrm{V}_0 = \pi^2\mathrm{R}^3$. We have $\kappa\rho_0 = 2/\mathrm{R}^2$, consequently the total mass of the world-matter is $\mathrm{M}_0 = 2\pi^2\mathrm{R}/\kappa$, or

$$\mathrm{M}_0 = 8 \,.\, 10^7\mathrm{R}, \qquad \rho_0 = \frac{8 \,.\, 10^6}{\mathrm{R}^2}.$$

If for M_0 we took the mass of our galactic system, which may be estimated † at $\frac{1}{3} \,.\, 10^{10}$, we would find $\mathrm{R} = 41$, which, of course, is absurd. A better estimate is found if we start from ρ_0 and take for this the star-density at the centre of the galactic system,‡ which may be estimated at about 80 stars in a unit of volume of Kapteyn (1000 cubic parsecs), or $\rho_0 = 10^{-17}$ in our units. We then find

(40) $$\mathrm{R} = 9 \,.\, 10^{11}.$$

* For the sake of comparison we may add that 10^6 astron. units = 5 parsecs = 16 light-years = $15 \,.\, 10^{18}$ centimetres. The mass of the sun is $2 \,.\, 10^{33}$ grams. A density 1 in astronomical units is therefore equivalent to $6 \,.\, 10^{-7}$ in C.G.S. units.

† Communicated by Professor Kapteyn. The estimate is based on van Rhyn's recent investigation of the number of stars (*Groningen Publications*, 27), assuming the average mass of the stars to be the same as that of the sun.

‡ In his paper of 1900, which is quoted below, Schwarzschild considers an elliptical universe just large enough to contain our galactic system (with a constant density equal to the star-density near the sun). This would be a very simple solution of the problem of what prevents the disintegration of the galactic system by the proper motions of the stars. The same argument is used by Einstein in the introductory paragraph of his *Kosmologische Betrachtungen*. It is evident, however, from the numerical data given above, that the elliptical space of the system A does *not* fulfil this purpose. In it the stars can escape from the galactic system quite as easily as in the classical euclidean space (system C).

The total mass would then be $M_0 = 7 \cdot 10^{19}$, and the volume $V_0 = 7 \cdot 10^{36}$.

It is very probable that in the part of space which immediately surrounds our galactic system there are many similar systems whose mutual distances are large compared with their dimensions. If for the average shortest distance between neighbouring systems we take 10^{10}, and if further we suppose that not only our neighbourhood but the whole universe is thus filled with galactic systems, there would be room for $7 \cdot 10^6$ of such systems. If each of these had a mass of $\frac{1}{3} \cdot 10^{10}$, their combined mass would be $2 \cdot 10^{16}$, or only $\frac{1}{3000}$ of the total mass of the world-matter. According to this view, only a small portion of the world-matter would be condensed into ordinary matter. It is, however, very well possible to imagine a world in which all the world-matter would be thus condensed. We must then, as unit of volume with which to measure the average density ρ_0, take a space which is large with respect to the mutual distances of the galactic systems. With the numerical data adopted above, we would then have $\rho_0 = \frac{1}{3} \cdot 10^{-20}$, from which

$$(41) \qquad R \leqq 5 \cdot 10^{13}.$$

I write the sign $\leq$ instead of $=$ because, if we took a still larger value for R, the total mass of the world-matter would not be sufficient to fill the universe with galactic systems. We can thus consider the value (41) as an upper limit—subject, of course, to the uncertainty (which is considerable) of the hypotheses and of the numerical data from which it was derived.

Space being finite, and the straight line closed, we should, at the point of the heavens opposite the sun, see an image of the back side of the sun. This not being the case, the light must be absorbed on its way "round the world." Schwarzschild* estimates that an absorption of 40 magnitudes would be sufficient. If we accept the result of Shapley,† that the absorption in intergalactic space is less than 0·0001 mag. in a Kapteyn's unit of distance (10 parsecs), then for an absorption of 40 mags. to be produced in a distance of πR we must have, in our units,

$$(42) \qquad R > \tfrac{1}{4} \cdot 10^{12}.$$

King‡ has derived the density of matter in space from the coefficient of selective absorption. The selective absorption found by Shapley is about one-fiftieth of the value used by King. The latter finds a density of 6300 suns per cubic parsec. Shapley's absorption would thus require one-fiftieth of this, or $\frac{3}{2} \cdot 10^{-14}$ in our units. This would correspond to $R = 2 \cdot 10^{10}$. With this value of R the total absorption in the distance πR would only be 3·6 mags., or only one-eleventh of the required value. To get an

* "Ueber das zulässige Krümmungsmaass des Raumes," *Vierteljahrssch. der Astr. Ges.*, vol. xxxv. (1900), p. 337.

† *Mount Wilson Contributions*, No. 116.

‡ *Nature*, vol. xcv. p. 701.

absorption of 40 mags., we must multiply the density by 11^2, and consequently divide R by 11. This would give

$$\rho_0 = 2 \cdot 10^{-12}, \qquad R = 2 \cdot 10^9.$$

This density appears to be much too large, and R much too small, to be admissible. The two assumptions underlying this determination are: 1st, that the coefficient of general absorption (extinction) is equal to that of selective absorption by molecular scattering; and 2nd, that the world-matter consists of molecules of hydrogen. Both assumptions may be considerably in error, and the extinction produced by a given density may well be much larger, and consequently the density needed for a given extinction much smaller, than has been here assumed.

Each of the estimates (39), (40), (41), (42) of course is subject to a very large uncertainty. Their near agreement is rather remarkable, and could not have been expected *a priori*.

7. In the system B the rays of light are straight lines in hyperbolical space. The parallax has, according to the formula (20′), a minimum $p_0 = 1/R$. Schwarzschild has, in the paper already quoted, derived a lower limit for the value of R of hyperbolical space, from the fact that there are certainly stars with parallaxes equal to 0″·05 or smaller. He thus found

$$R > 4 \cdot 10^7.$$

Of course only actually measured absolute parallaxes can be taken into account. All parallaxes measured after 1900 are relative ones, and consequently the limit found by Schwarzschild still corresponds to our present knowledge.

The reasoning by which the value (39) of R was derived for the system A is not applicable to the system B, for the relation between apparent and linear diameter is here

$$\delta = \frac{d}{R \sinh \dfrac{h}{R}} = \frac{d}{R \tan \chi},$$

which shows that δ is zero for $r = \frac{1}{2}\pi R$. If we accept the existence of a number of galactic systems whose average mutual distances are of the order of 10^{10}, all we can say is that πR must be several times 10^{10}, or roughly

$$R > 10^{11}. \tag{43}$$

Also the estimate of R, based on the fact that we do not see the back of the sun, is not applicable to the system B, because light requires an infinite time for the "voyage round the world."

In the system B we have $g_{44} = \cos^2 \chi$. Consequently the frequency of light-vibrations diminishes with increasing distance from the origin of co-ordinates. The lines in the spectra of very distant stars or nebulæ must therefore be systematically displaced towards the red, giving rise to a spurious positive radial velocity.

It is well known that the helium stars do indeed show a systematic displacement, corresponding to about $+4{\cdot}5$ km./sec. If we ascribe about one-third of this to the mass of the stars themselves,* the rest, or $+3$ km./sec., may be explained as an apparent displacement due to the diminution of g_{44}. For the average distance of the B-stars we can take † $r = \mathrm{R}\chi = 3 \cdot 10^7$. We then have $1 - \cos\chi = 10^{-5}$, from which

$$\mathrm{R} = \tfrac{2}{3} \cdot 10^{10}. \tag{44}$$

Campbell has also found a systematic displacement of the same sign for the K-stars, whose average distance probably is the largest after the helium stars. For stars of other types both the systematic displacement and the average distance are smaller.

For the lesser Magellanic cloud Hertzsprung found the distance $r > 6 \cdot 10^9$. The radial velocity‡ is about $+150$ km./sec. This gives

$$\mathrm{R} > 2 \cdot 10^{11}. \tag{45}$$

The formulas (25′), for small values of r, become the same as in classical mechanics. For large values of r there is no reason why the angular proper motion $\frac{d\theta}{dt}$ should not decrease in the same way as it does in Newtonian mechanics. The total linear velocity, however, and consequently also the radial velocity, may on the average be expected to increase up to $\chi = \frac{1}{4}\pi$, owing to the first term on the right in the second formula (25′). We should thus, in the system B, for stars in our neighbourhood expect radial and transversal velocities of the same order, but for objects at very large distances we should expect a greater number of large or very large radial velocities. Spiral nebulæ most probably are amongst the most distant objects we know. Recently a number of radial velocities of these nebulæ have been determined. The observations are still very uncertain, and conclusions drawn from them are liable to be premature. Of the following three nebulæ, the velocities have been determined by more than one observer: §

Andromeda	(3 observers)	− 311 km./sec.
N.G.C. 1068	(3 ,,)	+ 925 ,,
N.G.C. 4594	(2 ,,)	+ 1185 ,,

These velocities are very large indeed, compared with the usual velocities of stars in our neighbourhood.

The velocities due to inertia, according to the formula (25′), have no preference of sign. Superposed on these are, however, the apparent radial velocities due to the diminution of g_{44}, which are positive. The mean of the three observed radial velocities stated

* See first paper, p. 719.
† See *Astrophysical Journal*, vol. xxxii. p. 90.
‡ Report to the Council, 1917 Feb., *M.N.*, lxxvii. p. 376.
§ Report to the Council, 1917 Feb., *M.N.*, lxxvii. pp. 375, 383.

above is +600 km./sec. If for the average distance we take 10^5 parsecs $= 2 \cdot 10^{10}$, then we find

$$(46) \qquad R = 3 \cdot 10^{11}.$$

Of course this result, derived from only three nebulæ, has practically no value. If, however, continued observation should confirm the fact that the spiral nebulæ have systematically positive radial velocities, this would certainly be an indication to adopt the hypothesis B in preference to A. If it should turn out that no such systematic displacement of spectral lines towards the red exists, this could be interpreted either as showing A to be preferable to B, or as indicating a still larger value of R in the system B.

Doorn: 1917 *July.*

The Equations of Radiative Transfer of Energy.

By J. H. Jeans.

1. In a gaseous star it is probable that much more energy is transferred by radiation than by ordinary gaseous conduction, so that an accurate determination of the laws of radiative transfer is a necessary preliminary to many problems of stellar physics. In two recent papers * Professor Eddington has given an equation of transfer which I believe to be erroneous (at least when considered with reference to its application), and which has, I believe, led him to an erroneous result. In the present paper I have examined independently the laws of radiative transfer.

These laws, like those of ordinary conduction of heat, can of course be found in a general form independently of the special problem to which they are ultimately to be applied; we need not complicate the laws by introducing (as Professor Eddington does) the curvature of the star's figure.

Consider a medium arranged so that layers of equal temperature are normal to the axis of x, and consider the stream of radiation making an angle θ with this axis. At the plane $x = \xi$ let the stream of radiation crossing a plane of cross-section $d\sigma$ per unit time in directions contained within a small cone $d\omega$ be $I d\sigma d\omega$, where I is a function of x and θ.

After traversing a length of path ds, the main part of the radiation will reach the plane $x = \xi + ds \cos\theta$. On this path it will have been diminished in a ratio $1 - c\rho ds$, where c is a coefficient of opacity for this particular radiation, measured per unit mass, and ρ is the density of the medium. It will also have been reinforced by radiation emitted by the matter traversed. The volume of this matter is $ds d\sigma$, so that if E is the emission

* *M.N.*, 1916 Nov. and 1917 June.